Characters of Representations, Briefly

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Representations and Fundamental Results

What is a Representation?

Given a group G and a vector space V, a *representation* of G over V is a group homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

We focus now on the usual case of G being a finite group and V being a $\mathbb{C}$-vector space, although many results hold in further generality.

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Why do we care?

Representations allow us to study groups by studying matrices. In particular, a *faithful* representation is one where ρ is injective, giving us an isomorphic copy of G inside of a matrix group.

Examples

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For $G = \mathbb{Z}/3\mathbb{Z}$, we get a representation $\rho : G \to \mathbb{C}^\times$ by

$$\rho(0) = 1 \qquad \rho(1) = \omega \qquad \rho(2) = \omega^2$$

where ω is the primitive third root of unity, $\omega = e^{\frac{2\pi i}{3}}$. We can think of these as rotations of the complex plane; there is a similar representation for any cyclic group.

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If we let $\mathbb{C}G$ be a $\mathbb{C}$-vector space with basis indexed by the elements of G, we can define the *left regular representation* $\lambda : G \to \mathrm{GL}(\mathbb{C}G)$ by

$$[\lambda(g)]\left(\alpha_1 e_{g_1} + \cdots + \alpha_n e_{g_n}\right) = \alpha_1 \left(e_{gg_1}\right) + \cdots + \alpha_n \left(e_{gg_n}\right)$$

where $n = |G|$. We call $\mathbb{C}G$ the *group algebra* of G.

Convention

We notice that a representation $\rho : G \to \mathrm{GL}(V)$ gives us an action of G onto V by

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Building New Representations

Given representations V and W of G, we can form new representations:

- G acts on $V \oplus W$ by $g \cdot (v, w) = (g \cdot v, g \cdot w)$
- G acts on $V \otimes W$ by $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$

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Maschke's Theorem

Given a subrepresentation W of V, it has a complementary subrepresentation W' such that $V = W \oplus W'$. In this way, representations of finite groups are indecomposable exactly when they are irreducible, which is not true in generality.

Factoring Representations

These results lead to a major result in the representation theory of finite groups:

For any representation V of G we can write

$$V \simeq V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}$$

where each V_i is a distinct irreducible representation of G.

This means that studying the irreducible representations of G tells us how *every* representation of G acts.

Class Functions and Characters

Class Functions and Characters

Class Function

A *class function* is a function $\alpha : G \to \mathbb{C}$ that is constant on conjugacy classes, i.e., $\forall g, h \in G$

$$\alpha(g) = \alpha(hgh^{-1})$$

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The class functions on a group form a $\mathbb{C}$-vector space.

Inner Product

We can take the inner product of two class functions α and β by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

Under this inner product, the indicator functions for the conjugacy classes form an orthogonal basis for the space of class functions.

Characters

The *character* $\chi_V : G \to \mathbb{C}$ of a representation $\rho : G \to \mathrm{GL}(V)$ is defined as

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Using Characters

Since characters use the trace which is the sum of the eigenvalues, we get that

$$\chi_{V \oplus W} = \chi_V + \chi_W \qquad \chi_{V \otimes W} = \chi_V \chi_W \qquad \chi_{V^*} = \overline{\chi_V}$$

and since $\rho(e) = I$, we get that $\chi_V(e) = \dim(V)$.

Characters as Class Functions

Since the trace of a matrix satisfies $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we get that $\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(B)$, and thus our character is a class function:

$$\chi(hgh^{-1}) = \operatorname{tr}(\rho(hgh^{-1})) = \operatorname{tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \operatorname{tr}(\rho(g)) = \chi(g)$$

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Irreducibles form a Basis

The characters for the irreducible representations of G form an orthonormal basis for the space of class functions. This motivates us to study the distinct irreducible characters of G.

The Character Correspondence

Corollaries

Because the irreducible characters form a basis for the rest of the characters, we get several very important corollaries:

- Representations V and W are equivalent if and only if $\chi_V = \chi_W$.
- A representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.
- If V is an irreducible representation, the multiplicity of V in the decomposition of W is obtained by $\langle \chi_V, \chi_W \rangle$.

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Since the indicators functions for the conjugacy classes of G also form a basis for the space of class functions, the number of distinct irreducibles is in bijection with the number of conjugacy class of G, but there is no natural correspondence in general.

Properties of The Regular Representation

Recall $\lambda : G \to \mathrm{GL}(\mathbb{C}G)$ acts by $g \cdot e_h = e_{gh}$. This means that $\lambda(g)$ is a permutation matrix and the character χ is the number of fixed points, thus giving us

$$\chi(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

We then see that for an irreducible representation V,

$$\langle \chi_V, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi(g) = \overline{\chi_V(e)} = \dim(V)$$

Thus the decomposition of the regular representation gives two very important identities:

$$\mathbb{C}G \simeq V_1^{\oplus \dim(V_1)} \oplus \cdots \oplus V_k^{\oplus \dim(V_k)} \qquad \text{and} \qquad |G| = \chi(e) = \dim(\mathbb{C}G) = \sum (\dim V_i)^2$$

Character Tables

Column Invariance

This last result can be generalized, giving us the formula

$$\frac{1}{|G|}\sum \overline{\chi_{V_i}(g)}\chi_{V_i}(h) = \begin{cases} 1/c(g) & \text{if } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$$

where $c(g)$ is the order of the conjugacy class of g. In particular, this gives us

$$|G| = c(g)\sum \left|\chi_{V_i}(g)\right|^2$$

which can be seen visually in a character table.

Character Table

The *character table* for a group is a square table listing all of the conjugacy classes, irreducible representations, and the values of each character on each class.

Example

For $G = \mathbb{Z}/3\mathbb{Z}$, we see that since the group is abelian, each element has its own conjugacy class.

The two nontrivial representations are obtained by sending 1 to the primitive 3^{rd} roots of unity.

Since the conjugacy classes are in bijection with the irreducible representations, the table is square.

	1	1	1
$\mathbb{Z}/3\mathbb{Z}$	$[0]$	$[1]$	$[2]$
T	1	1	1
V_2	1	ω	ω^2
V_3	1	ω^2	ω

Example

For $G = A_4$, we take $N = \langle (12)(34), (14)(23) \rangle$ and see that $A_4/N \simeq \mathbb{Z}/3\mathbb{Z}$. Thus we can extend our previous representations of $\mathbb{Z}/3\mathbb{Z}$ to A_4 as such:

$$\rho : G \to G/N \to \mathrm{GL}(V)$$

	1	4	4	3
A_4	(1)	(123)	(132)	$(12)(34)$
T	1	1	1	1
V_2	1	ω	ω^2	1
V_3	1	ω^2	ω	1
V_4	3	0	0	-1

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We see that the first column then allows us to solve for the dimension of V_4 using

$$|A_4| = \sum (\dim V_i)^2 = \sum \chi_i(e)^2$$

and the column invariance further gives us

$$|A_4| = c(\sigma) \sum |\chi_i(\sigma)|^2$$

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Big Picture

Questions?

Final Overview

- Use *group representations* to study finite groups as matrices.
- Since all representations are fully decomposable, we only study the *irreducibles*.
- The irreducible *characters* form an orthonormal basis for class functions, giving us many nice properties.
- We can collect all of this information into a *character table*, allowing us to see the relationships easier.

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